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CFT and black hole entropy in induced gravity

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Abstract: We present a derivation of the entropy of black holes in induced gravity models based on conformal properties of induced gravity constituents near the horizon. The four-dimensional (4D) theory is first reduced to a tower of two-dimensional (2D) gravities such that each 2D theory is induced by fields with certain momentum p along the horizon. We demonstrate that in the vicinity of the horizon constituents of the 2D induced gravities are described by conformal field theories (CFT) with specific central charges depending on spin and non-minimal couplings and with specific correlation lengths depending on the masses of fields and on the momentum p. This enables one to use CFT methods to count partial entropies $s(p)$ in each 2D sector. The sum of partial entropies correctly reproduces the Bekenstein-Hawking entropy of the 4D induced gravity theory. Our results indicate that earlier attempts of the derivation of the entropy of black holes based on a near-horizon CFT may have a microscopic realization.

Keywords: Conformal Field Models in String Theory, Black Holes, Models of Quantum Gravity.

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Contents

1. Introduction

The problem of a microscopic explanation of the Bekenstein-Hawking entropy S^{BH} of black holes [1]–[3] was a subject of intensive investigations over the last ten years. This brought a number of very interesting results, especially in string theory where S^{BH} has been calculated by statistical methods for certain types of black holes [4] (for a review see e.g. [5, 6]). Despite of that progress the problem is still far from its complete resolution and attracts a lot of interest. The string theory has brought new stimulating ideas, one of which was a role of two-dimensional conformal theory (CFT) in the entropy counting.

The importance of the conformal symmetry was appreciated after the computation made by Strominger [7] who derived with its help the entropy of black holes in 2+1 dimensional asymptotically anti-de Sitter space-times [8]. This result was strongly related to the earlier work by Brown and Henneaux [9] who pointed out that the conformal algebra for such space-times had a central extension. Later Carlip [10]–[12] suggested a universal way how to represent S^{BH} in a statistical-mechanical form by using a conformal symmetry of the gravitational action near the black hole horizon (see also work by Solodukhin [13]). The idea was that a Poisson bracket realization of a near-horizon conformal symmetry yields

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a Virasoro algebra with a central charge c proportional to S^{BH}. The entropy then can be derived from the known expression for the degeneracy of states of CFT with the given central charge at a fixed energy.

The approach [10]–[12] gives a statistical *representation* of S^{BH}. It implies the existence of the corresponding microstates but says nothing about their actual physical meaning. Related to the fact that the microscopic theory is not known, there is a certain ambiguity in definition of the corresponding conformal theory. In particular, the derivation of the entropy requires introduction of an additional parameter l, the 'size of the system'. Unfortunately, l, which can be considered as a finite correlation length, cannot be fixed from general principles.

It would be interesting to understand how the ideas of [10]–[12] are realized in simplified models of quantum gravity where microscopic degrees of freedom are known. There is a class of such models known as the *induced gravity theory.* The idea that gravity can be induced by quantum effects of matter fields goes back to Sakharov's work [14] (for a recent review of this subject see [15]). The microscopic degrees of freedom of induced gravity are quantum fields responsible for generation of the Einstein term in the low energy effective gravitational action. These fields are called *constituents.* The mechanism of the entropy generation in the induced gravity is similar to the one proposed in [16, 17] where the Bekenstein-Hawking entropy was identified with the entanglement entropy of quantum fields in the black hole background. But there is one important difference. Instead of the entanglement of states of low energy physical fields, in the induced gravity the entropy is generated because of the entanglement of constituents [18]. This idea has been developed in [19, 20] where a wide class of induced gravity models was considered.

It should be emphasized that the induced gravity, at least in its present form, is not a fundamental theory of quantum gravity. It allows one to reproduce correctly tree-level results of the standard Einstein gravity, but it does not give finite results for higher loop quantities. Nevertheless in studying the black hole entropy problem the induced gravity can serve as a convenient phenomenological model. Its main characteristic properties are the following: (i) the low-energy gravity is an induced phenomenon, (ii) the underlying theory is free of one-loop ultraviolet divergences, and (iii) there exist microscopic degrees of freedom connected with the states of the constituents which play the role of internal degrees of freedom. One can expect that any complete self-consistent theory of quantum gravity must possess these properties. It is certainly true for the string theory which is now usually considered as candidate for this role. There are indications that the microscopic explanation of the black hole entropy in induced gravity may be similar to the origin of the entropy in open string theory [21]. Thus, without pretending to be as fundamental as the string theory, the induced gravity may provide us with simple tools for developing the intuition about the quantum gravity.

The constituents in models considered in [19, 20] are massive non-interacting scalar and spinor quantum fields ϕ_k. They propagate on a classical background with metric $g_{\mu\nu}$ and the action for $g_{\mu\nu}$ is defined as the effective action $\Gamma[g]$ of the theory,

$$e^{i\Gamma[g]} = \int [D\phi]\; e^{i\sum_k I[\phi_k,g]}\,, \tag{1.1}$$

where $I[\phi_k, g]$ is the action of the k-th field. The masses of constituents are assumed to be of the order of the Planck mass m_{Pl}. In the low energy limit when the curvature R of the background is much smaller than m_{Pl}^2, the gravitational action coincides with the Einstein-Hilbert action

$$\Gamma[g] \simeq \frac{1}{16\pi G} \int \sqrt{-g} d^4x R[g] \,. \tag{1.2}$$

To make the models self-consistent the parameters of the constituents are chosen to eliminate the cosmological constant and cancel ultraviolet divergences in the induced Newton constant G. To satisfy the latter requirement one has to assume that some scalar constituents are non-minimally coupled, i.e., include terms like $\xi R\phi^2$ in the lagrangian. In four-dimensional theories the constraints leave a logarithmic divergence in a term in the effective action which is quadratic in curvature. This, however, causes no problem for studying entropy of Schwarzschild and Kerr black holes which are Ricci flat.

Note that the local temperature, as measured by an observer at rest, infinitely grows near the black hole horizon. Therefore, the induced gravity constituents in the region close to the horizon are effectively massless and conformally invariant. In the present paper we investigate the role of conformal properties in derivation of S^{BH} in induced gravity. This enables one to test the idea of [10]–[12] on concrete models where the microscopic degrees of freedom are known. Our results suggest a possible answer to some questions appearing in [10]–[12], like, for example, the origin of a finite correlation length l. They also indicate that general arguments when applied to a specific theory may need considerable modification.

The paper is organized as follows. In section 2 we discuss general properties of induced gravity models and the dimensional reduction of these models. In particular, we show that after the dimensional reduction the induced gravity in the region near the black hole horizon is reduced to a set two-dimensional (2D) theories. These reduced theories describe constituents with a fixed momentum p along the horizon. The four-dimensional (4D) theory is recovered by summing over contributions of all momenta. In this representation the 4D induced constant G^{-1} is the sum of 2D gravitational couplings $G_2(p)$ at different momenta p. This observation enables one to study 2D induced gravities instead of 4D gravity.

Such theories are not conformally invariant. The conformal invariance is broken by the masses and non-minimal couplings. We analyze the role of these effects in section 3 and conclude that in the near-horizon region each constituent can be related to a conformal theory with some central charge and a finite correlation length. A non-minimally coupled field with coupling ξ results in a CFT with a central charge $1 - 6\xi$.

In section 4, by taking into account that at each momentum p we have a set of CFT's with different charges and different sizes we compute a "partial entropy" $s(p)$. The sum of $s(p)$ over all momenta reproduces the correct value of S^{BH}. The method of derivation of partial entropies is similar to computations in [10]–[12]. We finish the paper by summarizing our results and discussing remaining problems in section 5. We also discuss a possible relation between the black hole entropy and the information loss about the degrees of freedom under the Wilson renormalization group flow on the space of 4D induced gravity theories. Appendix A is devoted to properties of 2D induced gravities obtained under the dimensional reduction. In appendix B we show that our derivation of the entropy works for black holes in 3-dimensional space-times.

2. Induced gravity

2.1 Models and general properties

Induced gravity models may possess different types of the constituent fields. In order to make the consideration more concrete we use the special model suggested in [19]. However, most of the results of this paper can be easily generalized.

The model consists of N_s scalar constituents ϕ_s with masses m_s, some of the constituents being non-minimally coupled to the background curvature with corresponding couplings ξ_s, and N_d Dirac fields ψ_d with masses m_d. The corresponding actions in (1.1) are

$$I[\phi_s, g] = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\nabla\phi_s)^2 + \xi_s R\phi_s^2 + m_s^2\phi_s^2\right], \tag{2.1}$$

$$I[\psi_d, g] = \int d^4x\sqrt{-g}\;\bar{\psi}_d(i\gamma^\mu\nabla_\mu + m_d)\psi_d\,. \tag{2.2}$$

We impose the following constraints on parameters of the constituents:

$$p(0) = p(1) = p(2) = p'(2) = 0\,, \tag{2.3}$$

$$q(0) = q(1) = 0, \tag{2.4}$$

where

$$p(z) = \sum_s m_s^{2z} - 4\sum_d m_d^{2z}\,, \qquad q(z) = \sum_s m_s^{2z}(1-6\xi_s) + 2\sum_d m_d^{2z}\,. \tag{2.5}$$

Constraints (2.3) serve to eliminate the induced cosmological constant while conditions (2.4) enable one to get rid of the ultraviolet divergences in the induced Newton constant G. It is the second set of conditions that will be important for our analysis of black hole entropy. Given (2.4) G is defined by formula

$$\frac{1}{G} = \frac{1}{12\pi}q'(1) = \frac{1}{12\pi}\left[\sum_s(1-6\xi_s)m_s^2\ln m_s^2 + 2\sum_d m_d^2\ln m_d^2\right]. \tag{2.6}$$

The Bekenstein-Hawking entropy of a black hole with the horizon area $\mathcal{A}$ is given by the usual formula

$$S^{BH} = \frac{1}{4G}\mathcal{A} \tag{2.7}$$

with G defined in (2.6). Because G is explicitly known one can prove that

$$S^{BH} = S - Q\,. \tag{2.8}$$

Here S is a statistical-mechanical entropy of the constituents thermally distributed at the Hawking temperature in the vicinity of the horizon . The quantity Q is a quantum average of a Noether charge defined on the bifurcation surface of the horizon by Wald's method [22]. The appearance of Q is related to the fact that induced gravity models require non-minimally coupled constituents. Equation (2.8) is universal: it is valid for different models including those with vector constituents [23] as well as for different kinds of black holes, rotating [24] and charged [25], in different space-time dimensions.

The physical reason of subtracting Q in (2.8), as was explained in [20], is related to two inequivalent definitions of the energy in the black hole exterior. One definition, $\mathcal{H}$, is the canonical energy or the hamiltonian. The other definition, $\mathcal{E}$, is the energy expressed in terms of the stress-energy tensor $T_{\mu\nu}$ which is obtained by variation of the action over the metric tensor. The two energies correspond to different properties of a black hole. $\mathcal{H}$ corresponds to evolution of the system along the Killing time and for this reason the operator $\mathcal{H}$ in quantum theory is used for constructing the density matrix which yields the entropy S in (2.8). On the other hand, $\mathcal{E}$ is related to thermodynamical properties of a black hole. If the black hole mass measured at infinity is fixed the change of the entropy S^{BH} caused by the change of the energy $\mathcal{E}$ of fields in black hole exterior is

$$T_H \delta S^{BH} = -\delta \mathcal{E}\,, \tag{2.9}$$

where T_H is the Hawking temperature of a black hole. The reason why $\mathcal{E}$ and $\mathcal{H}$ are not equivalent is in the existence of the horizon. The two quantities being integrals of metrical and canonical stress tensors differ by a total derivative. This difference results in a surface term on the bifurcation surface of the horizon. This surface term is not vanishing because the horizon is not a real boundary and the only requirement for fields in this region is regularity. One can show [26] that the boundary term is the Noether charge Q appearing in (2.8). More precisely,

$$\mathcal{E} = \mathcal{H} - T_H Q\,. \tag{2.10}$$

According to (2.9) the black hole entropy is related with the distribution over the energies $\mathcal{E}$ of the induced gravity constituents. Hence, the subtraction of Q in (2.8) accounts for the difference between $\mathcal{E}$ and $\mathcal{H}$ in (2.10).

It should be noted, however, that an explicit calculation of the black hole degeneracy for a given mass M which is connected with the distribution of the constituent field states over the energies $\mathcal{E}$ is a problem. Two suggestions how it can be done are discussed in [20] and [27]. The difficulty is that in quantum theory a non-zero value of Q in (2.8) is ensured by the modes which, from the point of view of a Rindler observer, have vanishing frequencies, the so-called soft modes.

2.2 Dimensional reduction

In what follows we assume that the black hole mass is much larger that the Planckian mass and use a classical Schwarzschild metric to describe it. Moreover, since the black hole entropy is related to the behavior of the constituents near the horizon, the curvature effects can be neglected and we use the Rindler approximation. In this approximation the metric is

$$ds^2 = -\kappa^2\rho^2 dt^2 + d\rho^2 + dy_1^2 + dy_2^2\,, \tag{2.11}$$

where κ is the surface gravity and y_j are coordinates on the horizon. In what follows we first discuss the induced gravity theory on space-times with more general metrics

$$ds^2 = dl^2 + dy_1^2 + dy_2^2 \,, \tag{2.12}$$
$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta \,, \tag{2.13}$$

where $\gamma_{\alpha\beta}$ in (2.12) is a metric tensor of a 1+1 dimensional Lorenzian space-time $\mathcal{M}_2$. This generalization allows us to study conformal properties of the dimensionally reduced theories.

In this setting the dynamics of the constituents is essentially two-dimensional which can be easily seen if we use the Fourier decomposition in y-plane and define

$$\phi_{s,\mathbf{p}}(x) = \frac{1}{2\pi a}\int d^2y e^{-i\mathbf{py}}\phi_s(x,y)\,, \qquad \psi_{d,\mathbf{p}}(x) = \frac{1}{2\pi a}\int d^2y e^{-i\mathbf{py}}\psi_d(x,y)\,, \tag{2.14}$$

where $\mathbf{p}$ is a momentum along the horizon, $\mathbf{py} = p_i y^i$. To avoid volume divergences related to the infinite size of the horizon we assume that the range of coordinates y^i is restricted, $-a/2 \leq y^i \leq a/2$. This means that the horizon area $\mathcal{A}$ is finite and equal to a^2.

With these definitions actions (2.1), (2.2) on space-time (2.12) can be written in the form

$$I[\phi_s, g] = \sum_{\mathbf{p}} I[\phi_{s,\mathbf{p}}, \gamma]\,, \qquad I[\psi_d, g] = \sum_{\mathbf{p}} I[\psi_{d,\mathbf{p}}, \gamma]\,, \tag{2.15}$$
$$I[\phi_{s,\mathbf{p}}, \gamma] = -\frac{1}{2}\int d^2x\sqrt{-\gamma}\left[(\nabla\phi_{s,\mathbf{p}})^2 + \xi_s\mathcal{R}\,\phi_{s,\mathbf{p}}^2 + m_s^2(\mathbf{p})\phi_{s,\mathbf{p}}^2\right], \tag{2.16}$$
$$I[\psi_{d,\mathbf{p}}, \gamma] = \int d^2x\sqrt{-\gamma}\;\bar{\psi}_{d,\mathbf{p}}(i\gamma^\alpha\nabla_\alpha + m_d + \gamma^j p_j)\psi_{d,\mathbf{p}}\,, \tag{2.17}$$

where $\mathcal{R}$ and ∇'s are, respectively, the scalar curvature and covariant derivatives on $\mathcal{M}_2$ and

$$m_s^2(\mathbf{p}) = m_s^2 + \mathbf{p}^2\,. \tag{2.18}$$

The fact that for a spinor field $\psi_{d,\mathbf{p}}$ the mass term is described by a matrix $m_d + \gamma^j p_j$ does not create problem since the propagator in this theory has a pole at $m_d^2(\mathbf{p}) = m_d^2 + \mathbf{p}^2$. The quantity $m_k^2(\mathbf{p})$ (for $k = s$ or d) can be interpreted as the energy square of a particle with mass m_k and the transverse momentum $\mathbf{p}$. Thus, as follows from (2.15)–(2.17) each induced gravity constituent in the process of dimensional reduction produces a tower of 2D fields with masses equal to transverse energies of 4D fields.

Let us now discuss how to derive the induced Newton constant (2.6) in terms of parameters of two-dimensional fields. In induced gravity ultraviolet divergencies are not just truncated, but cancel themselves because of constraints. Therefore, in induced gravity the 4D effective action $\Gamma[g]$ is the sum of actions of dimensionally reduced models (see, e.g., [28]). So, on space-time (2.12)

$$\Gamma[g] = \sum_{\mathbf{p}} \Gamma_2[\gamma, |\mathbf{p}|] = \frac{a^2}{4\pi}\int_\sigma^\infty \Gamma_2[\gamma, p]dp^2\,. \tag{2.19}$$

Here $\Gamma_2[\gamma, p]$ is the effective action of two-dimensional gravity induced by constituents with the momentum $p = |\mathbf{p}|$. In the last equality in (2.19) we assumed that parameter a is large and replaced the sum over $\mathbf{p}$ by the integral over p. The coefficient $a^2/(4\pi)$ is related to the number of modes with momentum p^2 in an interval Δp^2. The lower integration limit σ in (2.19) is proportional to a^{-2} and is determined by the smallest momentum of a particle in the region of the horizon with area $\mathcal{A} = a^2$. At infinite a, σ vanishes.

The two-dimensional action can be easily calculated in the limit when the curvature $\mathcal{R}[\gamma]$ of 2D space-time with metric (2.13) is smaller than $m_k^2(\mathbf{p})$. One finds

$$\Gamma_2[\gamma, p] \simeq \frac{1}{4G_2(p)} \int \sqrt{-\gamma} d^2x \left(\mathcal{R} + 2\lambda_2(p)\right) . \tag{2.20}$$

Here $G_2(p)$ and $\lambda_2(p)$ are, respectively, the gravitational coupling and cosmological constant of the 2D theory,

$$\frac{1}{G_2(p)} = -\frac{1}{12\pi}\left[\sum_s (1-6\xi_s) \ln\left(m_s^2+p^2\right) + 2\sum_d \ln\left(m_d^2+p^2\right)\right], \tag{2.21}$$

$$\frac{\lambda_2(p)}{G_2(p)} = \frac{1}{4\pi}\left[\sum_s \left(m_s^2+p^2\right)\ln\left(m_s^2+p^2\right) - 4\sum_d \left(m_d^2+p^2\right)\ln\left(m_d^2+p^2\right)\right]. \tag{2.22}$$

Note that the same constraints which provide finiteness of the four-dimensional couplings also eliminate the divergences of $G_2(p)$ and $\lambda_2(p)$.

The four-dimensional Newton constant G can be found by summation over momenta in (2.19). One gets

$$\Gamma[g] = \frac{1}{16\pi G} \int \sqrt{-g} d^4x R[g], \tag{2.23}$$

where $R[g] = \mathcal{R}[\gamma]$ because the 4D action is considered on the space-time (2.12), (2.13). We also put here $a^2 = \int dy_1 dy_2$. The constant G is defined as

$$\frac{1}{G} = \lim_{p \to 0} \frac{1}{G(p)}, \qquad \frac{1}{G(p)} = \int_{p^2}^{\infty} \frac{d\tilde{p}^2}{G_2(\tilde{p})}, \tag{2.24}$$

and coincides with (2.6). Again, the integral in (2.24) is finite due to constraints (2.4).

2.3 Running gravitational couplings

Equation (2.24) can be considered as a formal representation of the induced Newton constant. In what follows, however, we would like to move further and treat the two-dimensional field models at any momentum p as real physical theories[1]. This, certainly, imposes further restrictions on parameters of the constituents dictated by physical requirements. Because (2.24) includes particles with arbitrary large momenta p we are forced to make some guesses about the theory beyond the Planckian energies. (Note that the contribution of momenta much larger than masses of all constituents asymptotically vanish

[1]It should be noted that the idea of two-dimensional reduction of quantum gravity at high energies is not new. It also appears under discussion of scattering of particles with energies at the center of mass comparable or larger than the Planckian energy, see [29, 30].

because of constraints.) It is natural to assume that at any p the 2D induced gravities have strictly positive gravitational couplings $G_2(p)$. As one of the arguments note that only in this case the Bekenstein-Hawking entropy of a back hole in such 2D theories is positive. This assumption is closely related to properties of the other parameter, $G(p)$, in (2.24). $G(p)$ has a meaning of a running Newton coupling constant in the theory with the ultraviolet cutoff at the transverse momentum p. To demonstrate this, let us suppose that all constituents have masses of the order of m_{Pl} and that the background curvature $|\mathcal{R}| \simeq m_{Pl}^2 + p^2$. Then the low-energy approximation (2.20) for induced actions is valid only for constituents with momenta larger than chosen value p. In this case the gravitational coupling of the four-dimensional theory is $G(p)$ because it is determined only by contributions of fields with momenta above p.

Interpretation of $G(p)$ as a running coupling also follows from the definition of the effective action in the Wilson renormalization group approach, see [31] (and section 12 of [32] for a good introduction). In standard Wilson approach the effective action at the scale p is defined by integrating in the functional integral over all modes with momenta larger than p (one uses the euclidean formulation). The distinction of our case is that the momentum is considered as the transverse one. After such integration one gets the induced gravity theory with constant $G(p)$ plus quantum theory of "low-energy" constituents with transverse momenta smaller than p.

If $G(p)$ is a running coupling, $G_2(p)$ is related to the beta-function of the theory because

$$\frac{1}{G_2(p)} = -\frac{\partial}{\partial p^2}\frac{1}{G(p)} \,. \tag{2.25}$$

When $G_2(p)$ is strictly positive, the four-dimensional coupling $G(p)$ monotonically increases and the gravity gets stronger and stronger at high energies. It is this sort of the behavior which one may expect in quantum gravity and it is what one finds in the Wilson renormalization group. We return to discussion of this point in the last section.

In appendix A we show that there is a class of induced gravity models with a softly broken supersymmetry where the cosmological constant $\lambda_2(p)$ is zero, while the gravitational coupling $G_2(p)$ is positive for all $p > 0$.

3. Conformal properties near the horizon

3.1 Effects related to masses

Let us discuss properties of two-dimensional fields $\phi_{s,\mathbf{p}}$, $\psi_{d,\mathbf{p}}$. If the fields were massless, and, additionally, $\phi_{s,\mathbf{p}}$ were minimally coupled, the classical actions (2.16), (2.17) would be conformally invariant. In quantum theory the conformal invariance would be broken by anomalies.

It is convenient to proceed further in the euclidean version of the theory by assuming that euclidean analog of $\mathcal{M}_2$ has a topology of a disk and the euclidean time τ (related to the Rindler time) is a periodic coordinate. The euclidean actions have the same form as Lorenzian functionals (2.16), (2.17) except that the scalar action has a different sign and factor i by the Dirac operator in (2.17) should be omitted. With definition of the stress-energy tensor

$$T_{\alpha\beta} = -\frac{2}{\sqrt{\gamma}}\frac{\delta I}{\delta \gamma^{\alpha\beta}}$$

one has

$$T_{\alpha\beta} = -\left(\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}\left((\nabla\phi)^2 + m^2 + \xi\tilde{R}\,\phi^2\right)\gamma_{\alpha\beta} + \xi\left(\gamma_{\alpha\beta}\nabla^2 - \nabla_\alpha\nabla_\beta\right)\phi^2\right), \quad (3.1)$$

for a 2D scalar constituent with a mass m (which depends on p) and a non-minimal coupling ξ. For a 2D spinor constituent (with or without mass)

$$T_{\alpha\beta} = \frac{1}{4}\left(\nabla_\alpha\tilde{\psi}\gamma_\beta\psi + \nabla_\beta\tilde{\psi}\gamma_\alpha\psi - \tilde{\psi}\gamma_\alpha\nabla_\beta\psi - \tilde{\psi}\gamma_\beta\nabla_\alpha\psi\right). \quad (3.2)$$

Here $\tilde{\psi}$ is an analog of conjugated spinor which in the euclidean theory is a variable independent of ψ.

Consider our theory on the Rindler space and make the Wick rotation. We obtain the euclidean theory on 2D space

$$dl^2 = \kappa^2\rho^2 d\tau^2 + d\rho^2 = dzd\bar{z}\,, \quad (3.3)$$

$0 \leq \tau \leq 2\pi/\kappa$ and $0 \leq \rho \leq \rho_+$ where ρ_+ is some radius. The complex coordinates are introduced as usually, $z = x_1 + ix_2$, where $x_1 = \rho\cos\kappa\tau$, $x_2 = \rho\sin\kappa\tau$. If the constituents are massless and minimally coupled the scalar action (2.16) takes form

$$I[\phi] = 2\int d^2z\ \partial\phi\bar{\partial}\phi \quad (3.4)$$

where $d^2z = dx_1dx_2$, $\partial = \partial/\partial z = \frac{1}{2}(\partial_1 - i\partial_2)$, $\bar{\partial} = \partial/\partial\bar{z}$. The integral (3.4) is invariant under the conformal transformation

$$z = f(w)\,, \qquad \bar{z} = \bar{f}(\bar{w})\,, \quad (3.5)$$

$$\phi(z,\bar{z}) = \phi'(w,\bar{w})\,. \quad (3.6)$$

To write transformations in the spinor theory it is convenient to define the components of spinors in (2.17) as follows $\psi^T = (\bar{b}_1, a_1, \bar{b}_2, a_2)$, $\tilde{\psi} = (b_1, \bar{a}_1, b_2, \bar{a}_2)$, where $\bar{b}_j, \bar{a}_j$ and b_j, a_j are independent variables. If we neglect the masses and transverse momenta the action (2.17) for a particular constituent has the form[2]

$$I[\psi] = \int d^2x\ \tilde{\psi}\nabla^\alpha\gamma_\alpha\psi = \sum_{j=1,2}\int d^2z\left[b_j\bar{\partial}a_j + \bar{b}_j\partial\bar{a}_j\right]. \quad (3.7)$$

It is invariant under conformal transformations of coordinates (3.5) when the spinor components transform as

$$b_j(z,\bar{z}) = (\partial f)^{-1/2}b'_j(w,\bar{w})\,, \qquad a_j(z,\bar{z}) = (\partial f)^{-1/2}a'_j(w,\bar{w})\,, \quad (3.8)$$

$$\bar{b}_j(z,\bar{z}) = (\bar{\partial}\bar{f})^{-1/2}\bar{b}'_j(w,\bar{w})\,, \qquad \bar{a}_j(z,\bar{z}) = (\bar{\partial}\bar{f})^{-1/2}\bar{a}'_j(w,\bar{w})\,, \quad (3.9)$$

[2]We use representation where $\gamma^\alpha = (\sigma^1 \times \sigma^\alpha)$, $\alpha = 1,2$ and σ's are the Pauli matrices.

where $\partial f = \partial_w f$. Let us consider now the holomorphic components of the stress-energy tensor[3], $T(z) = T_{zz}(z)$, for scalars and spinors, respectively,

$$T(z) = -(\partial\phi)^2 , \tag{3.10}$$
$$T_j(z) = \frac{1}{4}(\partial b_j a_j - b_j \partial a_j) . \tag{3.11}$$

(The anti-holomorphic components are defined analogously). It is well-known that in quantum theory the renormalized components transform as (see, e.g., [33])

$$T'(w) = (\partial f)^2 T(z) + \frac{1}{2\pi} A_f(w) , \tag{3.12}$$
$$T'_j(w) = (\partial f)^2 T_j(z) + \frac{1}{2\pi} A_f(w) , \tag{3.13}$$
$$A_f(w) = \frac{2\partial^3 f \partial f - 3(\partial^2 f)^2}{24(\partial f)^2} . \tag{3.14}$$

An anomalous term $A_f(w)$ appears in the transformation law because renormalization procedure requires subtracting divergent terms which are not conformally invariant. As follows from (3.12), (3.13), the conformal algebra in quantum theory has a central extension. The corresponding central charge c for each scalar field and each j-th component of the spinor field equals unity (thus the total central charge corresponding to a dimensionally reduced Dirac spinor is $c = 2$).

Let us now discuss what happens when fields are massive. Consider first a scalar field with a mass m. Consider the correlation function which determines the theory. In two dimensions

$$\langle \phi(z,\bar{z})\phi(z',\bar{z}')\rangle = \frac{1}{4\pi}\int_0^\infty \frac{ds}{s} e^{-\frac{d^2}{4s} - m^2 s} = \frac{1}{2\pi} K_0\left(d^2 m^2\right) , \tag{3.15}$$

where $d^2 = |z - z'|^2$ and $K_\nu(x)$ is the Bessel function. At small x the Bessel function has the following asymptotics

$$K_0(x) = -\ln\frac{x}{2} + O(x^2 \ln x) , \tag{3.16}$$

while at large x

$$K_0(x) = \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + O\left(x^{-1}\right)\right) . \tag{3.17}$$

Therefore, if the points $(z,\bar{z})$ and $(z',\bar{z}')$ are entirely inside the circle of the radius $R < m^{-1}$ the correlator is determined by the leading term in (3.16),

$$\langle \phi(z,\bar{z})\phi(z',\bar{z}')\rangle = -\frac{1}{4\pi}\ln\left(m^2|z - z'|^2\right) , \tag{3.18}$$

and it coincides up to an irrelevant constant with the correlator of the massless field. On the other hand, if one of the points is inside of the circle and the other is far outside, or the two points are outside and far apart from each other the correlator is exponentially small, see (3.17).

[3]The commonly used definition of $T(z)$ differs from our definition by the factor 2π.

This analysis shows that if we are interested in the behavior of the system inside the circle with radius m^{-1} we can consider it as a massless theory. In general, the mass breaks conformal invariance explicitly, but inside this circle the effect of the mass is small and theory is conformal. The conformal properties improve when one shrinks the circle toward its center. The only feature related to the fact that our theory is obtained as a result of the dimensional reduction is that mass is a function of the transverse momentum p. Thus, the radius is $R \sim (m^2+p^2)^{-1/2}$ and the larger the momentum p the smaller the region where theory is conformal.

Let us now discuss the case of spinor fields. If we keep mass and momentum terms the reduced action for a spinor with a momentum p has the form[4]

$$\begin{aligned} I[\psi] &= \int d^2x \, \tilde{\psi}(\nabla^\alpha \gamma_\alpha \psi - ip_j\gamma^j + im)\psi \\ &= \int d^2z \Bigg[\sum_{j=1,2} \left(b_j \bar{\partial} a_j + \bar{b}_j \partial \bar{a}_j + \frac{im}{2}\left(\bar{a}_j a_j + \bar{b}_j b_j\right)\right) + \\ &\qquad + \frac{\bar{\mu}}{2}(b_1\bar{b}_2 - \bar{a}_2 a_1) + \frac{\mu}{2}(\bar{a}_1 a_2 - b_2\bar{b}_1) \Bigg], \end{aligned} \tag{3.19}$$

where $\mu = \mu(\mathbf{p}) = p_1 + ip_2$, $\bar{\mu} = p_1 - ip_2$. In conformal theory (3.7) the only nonzero correlators are $\langle b_j a_j \rangle$ and $\langle \bar{b}_j \bar{a}_j \rangle$. In theory (3.19) new correlators appear. All of them are easy to calculate. For instance, for the field b_1 there are three non-trivial correlators:

$$\langle b_1(z,\bar{z}) a_1(z',\bar{z}') \rangle = -\frac{1}{\pi}\partial \, K_0\left(d^2 m^2(\mathbf{p})\right), \tag{3.20}$$

$$\langle b_1(z,\bar{z}) \bar{b}_1(z',\bar{z}') \rangle = -\frac{im}{2\pi} \, K_0\left(d^2 m^2(\mathbf{p})\right), \tag{3.21}$$

$$\langle b_1(z,\bar{z}) \bar{b}_2(z',\bar{z}') \rangle = \frac{\mu(\mathbf{p})}{2\pi} \, K_0\left(d^2 m^2(\mathbf{p})\right). \tag{3.22}$$

where $d^2 = |z-z'|^2$. Suppose now that both points are inside the circle with radius $R \sim |m(\mathbf{p})|^{-1}$. Then the correlator can be approximated by that of the conformal theory, see (3.18). In this case $\langle b_1 a_1 \rangle$ is the correlator of conformal fermions. Other correlators (3.21), (3.22) are not zero in this limit. However, they are smaller compared to $\langle b_1 a_1 \rangle$. Indeed, one can see that $\langle b_1 a_1 \rangle \sim (z-z')^{-1}$, while $\langle b_1 \bar{b}_1 \rangle \sim m \ln|z-z'|$, $\langle b_1 \bar{b}_2 \rangle \sim \mu \ln|z-z'|$. Thus, the new correlators can be neglected once the points are inside the circle and in this region fermions are well described by a conformal theory.

3.2 Rindler entropy in 2D theory

Let us discuss the entropy s of a massless scalar field in a two-dimensional Rindler spacetime (see (2.11))

$$dl^2 = -\kappa^2 \rho^2 dt^2 + d\rho^2 \,. \tag{3.23}$$

[4]We put $\gamma_{i=1} = (-i\sigma_2 \times I_2)$ (where I_2 is the unit matrix) and $\gamma_{i=2} = (\sigma_1 \times \sigma_3)$.

We define s as the entropy of Rindler quanta in some range of coordinate ρ, $\epsilon \leq \rho \leq R$. For quanta at the temperature $T_H = \frac{\kappa}{2\pi}$ the entropy is known to be

$$s = \frac{1}{6} \ln \frac{R}{\epsilon} \,. \tag{3.24}$$

If we have c fields

$$s\left(c, \frac{R}{\epsilon}\right) = \frac{c}{6} \ln \frac{R}{\epsilon} \,. \tag{3.25}$$

Formula (3.25) also holds for an arbitrary CFT if constant c in it is identified with the corresponding central charge. Such a result can be obtained by using conformal properties of the free energy and pointing out that c is related to the conformal anomaly [34].

The entropy of Rindler particles can be also interpreted as the entropy of entanglement between quantum states inside and outside the horizon. It is important to note that this entropy can be derived using the well-known Cardy formula, in a way which is parallel to the derivation of black hole entropy in Carlip's approach [10]–[12]. If the central charge c is fixed, $s(c, R/\epsilon)$ determines the degeneracy of operators L_0, $\bar{L}_0$ (which generate translations along coordinates $u = t - x$, $v = t + x$) in the state which corresponds to the Minkowski vacuum. We do not dwell here on how the Cardy formula can be used to obtain (3.25), the details can be found, e.g., in [35].

It follows from discussion of the preceding section that result (3.25) should not change if the field has a non-zero mass m provided that R is smaller than the corresponding correlation length, $R < m^{-1}$. This is easy to see by examining the equation for a Rindler mode $\phi_\omega(t, \rho) = e^{-i\omega t}\phi_\omega(\rho)$ with frequency ω,

$$\left(\left(\kappa^{-1}\omega\right)^2 - H^2\right) \phi_\omega = 0 \,, \tag{3.26}$$

$$H^2 = -(\rho\partial_\rho)^2 + \rho^2 m^2 \,. \tag{3.27}$$

The operator H is a single-particle hamiltonian. According to (3.26), the closer particle to the Rindler horizon $\rho = 0$ the smaller the effect of its mass. The mass becomes important when $\rho \sim m^{-1}$.

By taking into account that the normalized single-particle mode is expressed in terms of the Bessel function,

$$\phi_\omega(\rho) = \frac{\sqrt{\sinh(\pi\omega/\kappa)}}{\kappa\pi^2} K_{i\omega/\kappa}(m\rho) \,, \tag{3.28}$$

the free energy of the field in the region $\epsilon < \rho < R$ can be written in the form

$$F(\beta, \epsilon, R) = \beta^{-1} \int_0^\infty \Phi(\omega, \epsilon, R) \, \ln\left(1 - e^{-\beta\omega}\right) d\omega \,, \tag{3.29}$$

$$\Phi(\omega, \epsilon, R) = 2\omega \int_\epsilon^R \frac{d\rho}{\rho} |\phi_\omega(\rho)|^2 \,. \tag{3.30}$$

The quantity $\Phi(\omega, \epsilon, R)d\omega$ is the number of energy levels in the interval $(\omega, \omega + d\omega)$. If $R < m^{-1}$ and $\epsilon \to 0$ the spectral density (3.30) coincides with the spectral density of the massless theory in the same region $\epsilon < \rho < R$. However, for $R > m^{-1}$ the contribution from the domain $\rho > m^{-1}$ to the integral in (3.30) is exponentially suppressed due to the behavior of the Bessel function. This means that for $R > m^{-1}$ in the leading asymptotic the free energy coincides with the free energy of massless theory in the region $\epsilon < \rho < m^{-1}$. Such a conclusion is supported by the direct calculation which shows that for $\epsilon \ll m^{-1} \ll R$

$$\Phi(\omega, \epsilon, R) = -\frac{1}{\pi\kappa} \ln \frac{m\epsilon}{2} + f(\omega, \epsilon) + \bar{f}(\omega, \epsilon) \ , \tag{3.31}$$

$$f(\omega, \epsilon) = \frac{i}{4\pi\omega} \frac{\Gamma(1 + i\omega/\kappa)}{\Gamma(1 - i\omega/\kappa)} \left(\frac{m\epsilon}{2}\right)^{-2i\omega/\kappa} - \frac{1}{2\pi\kappa} \psi\left(1 - \frac{i\omega}{\kappa}\right) , \tag{3.32}$$

where $\bar{f}$ denotes complex conjugated function and $\psi(x)$ is the logarithmic derivative of the Γ function $\Gamma(x)$. In the limit $\epsilon \to 0$ the first term in the r.h.s. in (3.31) dominates over the last two ones. Thus, leaving in the spectral density only the logarithmic term one gets from (3.29)

$$F(\beta, \epsilon, R) = -\frac{\pi}{6\beta^2\kappa} \ln \frac{1}{m\epsilon} \, . \tag{3.33}$$

The corresponding entropy computed at the Hawking temperature $\beta^{-1} = \kappa/2\pi$ is

$$s\left(\frac{R}{\epsilon}\right) = \frac{1}{6} \ln \frac{1}{m\epsilon} \, , \tag{3.34}$$

as was expected.

3.3 Effects related to non-minimal couplings

So far we discussed the effects of massive terms. In addition to them there are other terms which break conformal invariance. In our theory these are non-minimal couplings in the scalar action (2.16). These terms modify the scalar field stress-energy tensor and its transformation properties. Instead of (3.10), (3.12) in case of non-minimal couplings one has (see, e.g., [36])

$$T(z) = -(\partial\phi)^2 + 2\xi((\partial\phi)^2 + \phi\partial^2\phi) \ , \tag{3.35}$$

$$T'(w) = (\partial f)^2 T(z) + (1 - 6\xi)\frac{1}{2\pi} A_f(w) - \xi\frac{1}{8\pi}(\partial \ln \partial f)^2 \, , \tag{3.36}$$

where we neglected by mass terms. Without the last term in r.h.s. of (3.36) this transformation law corresponds to a conformal theory with an effective central charge $c = 1 - 6\xi$.

Let us discuss conditions when the last term can be ignored. It is easy to see that this can be done if the conformal transformations are infinitesimal,

$$z = f(w) = w + \varepsilon(w) \, , \qquad |\varepsilon(w)| \ll 1 \, . \tag{3.37}$$

In this case variation of the stress-energy tensor under transformation (3.36) in the leading order in ε is

$$\delta T(w) = T'(w) - T(w) = \varepsilon(w)\partial T(w) + 2\partial\varepsilon(w) T(w) + \frac{1 - 6\xi}{24\pi}\partial^3\varepsilon(w) + O(\varepsilon^2) \, . \tag{3.38}$$

The last term in (3.36) gives contribution proportional to $(\partial\varepsilon)^2$ which is not important at this order.

Note that it is the linear order of the variations of the conformal stress-energy tensor which is needed to infer the structure of the Virasoro algebra. Therefore, when infinitesimal conformal transformations are considered, a 2D scalar field ϕ with a non-minimal coupling ξ is equivalent to a conformal theory with the effective central charge $c = 1 - 6\xi$. The concrete realization of this effective conformal theory in terms of the initial field ϕ may be non-trivial[5] but it is not important for the computation of the entropy. What really matters is the value of the central charge which enables one to fix uniquely the form of the free energy by using conformal properties (the conformal anomaly) of the effective action [34] and use formula (3.25) for the entropy.

The relevance of infinitesimal conformal transformations for computation of the entropy can be also seen from the following reasonings [36]. In the euclidean theory the entropy can be derived in a geometrical way by using the conical singularity method. To this aim one has to consider the transformation $f(w) = w^{\alpha}$ with $\alpha = \frac{2\pi}{\beta}$. Then in $(w, \bar{w})$–plane there is a conical singularity with the deficit angle $2\pi - \beta$. The parameter β corresponds to an inverse temperature and the limit $\beta = 2\pi$ corresponds to the Hawking temperature, or to a Minkowski vacuum in case of the Rindler space-time. To derive the entropy one has to take the euclidean action off-shell, find its derivative over β and after that go to limit $\beta = 2\pi$. It is easy to see that at small angle deficits $A_f(w) \sim (2\pi - \beta)$ and the corresponding term in (3.36) is related to contribution to the entropy proportional to the effective central charge $c = 1 - 6\xi$. As for the last term, its contribution vanishes because $(\partial \ln \partial f)^2 \sim (2\pi - \beta)^2$.

3.4 Negative central charge contributions

The constraints of induced gravity models require that at least some of effective central charges $c = 1 - 6\xi$ were negative. For $c < 0$ formula (3.25) results in a negative entropy. Typically CFT's with negative central charges correspond to ghosts. The ghosts appear in gauge theories when the Hilbert space is enlarged during the quantization. The ghosts do have negative entropy because it should compensate contribution of the extra degrees of freedom in the enlarged Hilbert space. However, if the system is unitary its total entropy is always positive.

There is some similarity between induced gravity models and gauge theories in the following sense [20]. As we discussed in section 2.1, the non-minimal couplings result in modification of the energy of the system $\mathcal{E}$ (adding a surface term $T_H Q$ at the horizon) but do not change the canonical energy $\mathcal{H}$, see (2.10). In quantum theory the value of Q is determined by the soft modes, i.e., modes with vanishing frequencies. The soft modes are analogous to pure gauge degrees of freedom because one can add an arbitrary number of these modes without changing the canonical energy $\mathcal{H}$. However, it is the energy $\mathcal{E}$ which is the physically observable quantity and, if $\mathcal{E}$ is fixed, then for given $\mathcal{H}$ the number of the soft modes cannot be arbitrary.

[5]It should be noted that the generator of the conformal transformation δT in (3.38) is not the operator T itself but only its part which does not depend on the parameter ξ.

In what follows we interpret negative contributions of some of the non-minimally coupled constituents to the entropy as the effect of such ghost-like degrees of freedom. We adopt computations of conformal field theories with central charges $c = 1 - 6\xi$ in order to obtain the distribution over the physical energies $\mathcal{E}$ instead of the canonical distribution. Note that, like gauge theories, the induced gravity models are unitary theories. The difference between $\mathcal{E}$ and $\mathcal{H}$ appears only if a Cauchy surface is divided by the horizon into "observable" and "non-observable" regions. This difference is important in the entropy problem since the entropy of the black hole itself is connected with the entanglement generated by the existence of the horizon. On the entire Cauchy surface, $\mathcal{E}$ and $\mathcal{H}$ coincide.

4. Counting the black hole entropy

The above arguments lead to the following conjecture: i) Each induced gravity constituent with the momentum p and mass m_k corresponds to a 2D conformal theory with a central charge c_k and a finite correlation length $R(p) = |m_k(\mathbf{p})|^{-1}$, $p = |\mathbf{p}|$; charges of spinor constituents are $c_d = 2$, while charges of scalar fields are $c_s = 1 - 6\xi_s$ and depend on non-minimal couplings. ii) Each constituent yields a contribution to the total entropy equal to

$$s\left(c_k, \frac{R_k(p)}{\epsilon}\right) = \frac{c_k}{6} \ln \frac{R_k(p)}{\epsilon}\,, \tag{4.1}$$

where ϵ is some cutoff near the horizon, universal for all fields. Equation (4.1) follows from (3.34).

To proceed it is convenient to represent the induced gravity constraints, which eliminate divergences in the Newton constant (see (2.4)), in the form

$$C = \sum_s c_s + \sum_d c_d\,, \qquad C = 0\,, \tag{4.2}$$

$$\sum_s c_s m_s^2 + \sum_d c_d m_d^2 = 0\,. \tag{4.3}$$

Constant C can be interpreted as a total central charge of constituents. The charge is zero because at each momentum p the 2D theory is free from ultraviolet divergences. Let us repeat that conditions (4.2), (4.3) are possible only when some scalar central charges c_s are negative.

Now it is easy to see that the entropy of all constituents in 2D induced gravity at some momentum p is

$$s(p) = \sum_k s\left(c_k, \frac{R_k(p)}{\epsilon}\right) = \frac{1}{6}\sum_k c_k \ln R_k(p) = \frac{\pi}{G_2(p)}\,, \tag{4.4}$$

where $G_2(p)$ is the 2D induced Newton constant defined in (2.21). The dependence on cutoff ϵ disappears because of (4.2). Note that we consider the models with strictly positive couplings $G_2(p)$ and, therefore, the partial entropy $s(p) > 0$, despite of contributions of some negative central charges. One can also see that for a black hole in 2D induced gravities (2.20) the entropy (4.4) coincides with the Bekenstein-Hawking entropy. Furthermore, the entropy of 4D theory, which is

$$S=\frac{a^2}{4\pi}\int_0^\infty s(p)dp^2=\frac{\mathcal{A}}{4G}\,, \tag{4.5}$$

coincides with the Bekenstein-Hawking entropy (2.7) of a four-dimensional black hole with the horizon area $\mathcal{A}=a^2$. The last equality in (4.5) follows from relation (2.24) between 4D and 2D couplings.

Therefore, entropy of induced gravity constituents computed in the near-horizon limit as the entropy of 2D conformal fields does reproduce the Bekenstein-Hawking entropy. This indicates that the approach by Carlip [10]–[12] may have realization in quantum gravity at the microscopic level.

The result (4.5) can be also obtained in a slightly different way. Let us first find the total entropy S_k of each constituent by integrating over the transverse momentum

$$S_k=\frac{a^2}{4\pi}\int_0^\Lambda s\left(c_k,\frac{R_k(p)}{\epsilon}\right)dp^2=c_k\frac{\mathcal{A}}{48\pi}\left(m_k^2\ln\frac{m_k^2}{\Lambda}+\Lambda(1-\ln\epsilon^2\Lambda)\right), \tag{4.6}$$

where Λ is a upper cutoff on the momentum $\mathbf{p}^2$, and then take the sum of entropies of all constituents

$$S=\sum_k S_k=\frac{\mathcal{A}}{48\pi}\left(\sum_k c_k m_k^2\ln m_k^2-\ln\Lambda\sum_k c_k m_k^2+\Lambda(1-\ln\epsilon^2\Lambda)\sum_k c_k\right)=S^{BH}\,. \tag{4.7}$$

Again the final result is finite and does not depend on the cutoff Λ because of constraints (4.2), (4.3).

Our last comment concerns the energy associated with the entropy of constituents. Fixing the energy is important for approach [10]–[12] because application of Cardy formula requires to know the values of generators L_0 and $\bar{L}_0$ whose degeneracy is studied. The energy of each 2D constituent with momentum p can be found from the corresponding free energy (3.33). For theory with a central charge c_k one finds

$$E\left(c_k,\frac{R_k(p)}{\epsilon}\right)=\partial_\beta\left(\beta F\left(\beta,c_k,\frac{R_k(p)}{\epsilon}\right)\right)\Big|_{\beta=\beta_H}\,. \tag{4.8}$$

The result taken at the Hawking temperature (corresponding to $\beta_H^{-1}=\kappa/(2\pi)$) is

$$E\left(c_k,\frac{R_k(p)}{\epsilon}\right)=\frac{1}{2}T_H\,s\left(c_k,\frac{R_k(p)}{\epsilon}\right) \tag{4.9}$$

and the total 2D and 4D energies are

$$E(p)=\sum_k E\left(c_k,\frac{R_k(p)}{\epsilon}\right)=\frac{1}{2}T_H s(p) \tag{4.10}$$

$$E=\frac{a^2}{4\pi}\int E(p)dp^2=\frac{1}{2}T_H S^{BH}\,. \tag{4.11}$$

So, the energy is proportional to the Bekenstein-Hawking entropy. Let us emphasize that E in (4.11) is not related to the mass of a black hole. The reason is that E, by the construction, corresponds to thermal excitations of Rindler-like quanta. However, there is also the energy of zero-point fluctuations of these quanta which in the Minkowski vacuum (for the Rindler approximation) compensates the energy E exactly. Thus, the physical meaning of E requires additional analysis.

5. Discussion

The aim of this paper was to investigate a role of the near-horizon conformal symmetry in calculating the Bekenstein-Hawking entropy S^{BH} in induced gravity models. The arguments that there is a universal way to represent S^{BH} in a statistical-mechanical form by using a near-horizon CFT were given in [10]–[13]. Being very general these arguments do not take into account any microscopic structure of the theory. Therefore, the question of whether the above approach can have a microscopic realization remains an open problem. It is that issue which has motivated our work. The microscopic degrees of freedom of induced gravity models are free quantum fields obeying certain constraints. So it is interesting to see how the ideas of [10]–[13] may or may not work in this simple case.

We have demonstrated that induced gravity models in a region near the horizon can be dimensionally reduced in such a way that each 4D constituent field results in a tower of 2D quantum theories defined in a 2D plane orthogonal to the horizon. Close to the horizon the 2D fields are effectively massless and under coordinate changes their stress-energy tensors transform as stress tensors of 2D conformal theories, each with its own central charge. This enables one to count the states of 2D induced gravities by using standard CFT methods. Summation of all partial entropies $s(p)$ reproduces the correct value of S^{BH}. By its nature, this method is very close to the idea of [10]–[13]. So one can say that the latter does have a realization in the induced gravity models.

However, several important features are missing in the previous works. Our first observation is that the total central charge in near-horizon CFT in 2D induced gravities at each transverse momentum p vanishes. This property just reflects the absence of the ultraviolet divergences in such theories. The second observation is that a non-zero value of a partial entropy $s(p)$ is a result of breaking the conformal symmetry. The reason is that the constituents are massive fields and CFT's are characterized by finite correlation lengths. Moreover, because masses of fields do not coincide in general, there is a variety of correlation lengths in the theory.

It is this feature which makes our computation distinct from the approach of [10]–[13]. A CFT in [10]–[13] is characterized by a single non-trivial central charge c and a single correlation length l. There is a freedom in choosing l or c. If l is fixed then c is fixed too. The induced gravity suggests that, perhaps, l is not a free parameter but a scale dynamically determined by quantum gravity effects at Planckian energies.

Our discussion leaves a number of open problems. One of them is the physical meaning of negative central charges which appear due to non-minimally coupled scalar constituents. Although we have argued that each partial entropy $s(p)$ is positive, there is a problem how to interpret the negative contributions to this quantity. One of the possibilities mentioned in section 3.3 is that negative charge CFT's correspond to ghost-like degrees of freedom.

Such degrees of freedom may appear on a part of the Cauchy hypersurface when it is divided by the horizon but it should not violate unitarity of the theory. It is fair to say that further work on this issue is needed.

In this paper we have reduced 4D induced gravity to a set of two-dimensional theories which we considered as fully physical ones. The nice feature of 2D theories is a possibility to apply Zamolodchikov's c-theorem [37, 33]. The theorem concerns an information loss under renormalization transformations in the direction of larger scales. It is interesting to search for possible relation between this information loss and the entropy of a black hole in induced gravity.

To this aim let us return to results of sections 3.1, 3.2 and consider a quantum scalar field of mass m on a flat 2D space-time. Suppose we study a correlator of this field (3.15) in the Rindler coordinates (3.3). Put one point in the correlator on the horizon $\rho = 0$ and the other point at some nonzero value $\rho = R$. We have two regimes: if $R \ll m^{-1}$ the correlator behaves as in massless theory, if $R \gg m^{-1}$ it is exponentially suppressed by the mass. One can interpret the evolution along the Rindler radius R as a renormalization group flow [33] between the two fixed points, the ultraviolet point (UV), $R = 0$, and infrared point (IR), $R \to \infty$. These asymptotic limits of the massive theory are described by UV and IR scale-invariant theories which lie at the end points of renormalization group trajectory, see, e.g., [38]. The UV central charge is $c_{UV} = 1$ and it coincides with the central charge of 2D massless scalar field theory, while the IR central charge c_{IR} vanishes.

Zamlodochikov's c-theorem enables one to construct a c-function $C(R,m)$ which is a combination of local two-point correlators of components of the renormalized stress-energy tensor of the field ϕ, such that $C(R,m)$ is non-increasing along renormalization group trajectory, $\partial_R C(R,m) \leq 0$, see the details in [33, 38]. At the fixed points the c-function coincides with the corresponding central charges. The mass of the field can be interpreted as a coupling constant while the mass term in the action near the fixed point $R = 0$ can be considered as a perturbation. Zamolodchikov's theorem indicates that there should exist some kind of entropy which measures the loss of information about the degrees of freedom of the theory whose wave length is smaller than the cutoff R.

In section 4 we defined a partial entropy $s(p)$ of constituents in 2D induced gravity as a sum of contributions from different fields. Because some fields correspond to CFT's with negative central charge the application of the c-theorem to each individual field is not justified because there is a problem with unitarity. The 2D fields at a given p should be considered only as a whole system. Such system is well-defined (free of ultraviolet divergences) and can be unitary. Note that transformations changing the length scale of the theory are equivalent to redefinition of transverse energies of fields $m(\mathbf{p})$. This is easy to see by examining correlators of massive fields (3.15). However, if one considers trivial transformations $m'(\mathbf{p}) = \alpha m(\mathbf{p})$ the partial entropy $s(p)$ and 2D gravitational coupling $G_2(p)$ do not change, see (4.4).

To have an analog of a renormalization flow in the induced gravity models one can choose another option. Let us consider the momentum p as a renormalization parameter. This makes sense for several reasons. First, decreasing p means decreasing transverse energies and is equivalent to going to larger length scales. Second, by changing p one

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moves from one 2D (or 4D) induced gravity to another theory in a consistent way without violating the induced gravity constraints (2.3), (2.4). Third, as we discussed in the end of section 2.2, the parameter p becomes a real ultraviolet cutoff when the induced gravity theory is considered on space-times (2.12), (2.13) with 2D curvature $\mathcal{R}$ larger than $m_{Pl}^2+p^2$.

Identification of p with the renormalization parameter has interesting consequences. In the ultraviolet limit, when $p \gg m_{Pl}$, 4D and 2D running gravitational couplings, $G(p)$, $G_2(p)$, are large. On the other hand, the partial entropy $s(p)$ and the Bekenstein-Hawking entropy vanish, in accord with the fact that no information is lost in UV limit. In the infrared limit, $p \to 0$, the couplings are finite and the value of $G(p)$ approaches the value of the Newton constant for 4D induced gravity given by (2.6). The loss of the information in the IR limit is maximal and it can be measured by the Bekenstein-Hawking entropy. The coupling $G_2(p)$ can be related to the β-function of the theory, see (2.25). If $G_2(p) > 0$ the 4D running coupling $G(p)$ is monotonically increasing when one moves from UV to IR points. Therefore, the parameter $G^{-1}(p)$ behaves as a c-function.

To summarize, using the induced gravity as a toy model one can obtain an interesting information concerning the mechanism of the entropy generation in black holes. The entanglement of the constituents in the presence of the horizon plays a fundamental role in this mechanism. The universality of the Bekenstein-Hawking entropy, that is its independence of the detailed structure of the background microscopic theory of constituents which induce the gravitational field, is a natural property of the induced gravity models. In this paper we demonstrated that the induced gravity models allow one to exploit the results of the CFT for the explanation of black hole entropy. This can be used to check different ideas proposed in this area by the string theory. We are fully aware of the remaining difficulties which indicate that even in such simplified models the statistical-mechanical calculation of the Bekenstein-Hawking entropy is a very non-trivial task.

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A. Examples of induced gravity models with steady RG flow

Explicit examples of induced gravity models with two-dimensional constant $G_2(p)$ positive for all $p > 0$ is easy to construct. Not to have problems with induced cosmological constant consider a version of induced gravity where all constituents are combined in supersymmetric multiplets. Each multiplet consists of one Dirac spinor and 4 real scalar fields. The masses of all fields entering the same multiplet coincide. Also non-minimal coupling constants of scalar components insides the multiplets are the same. Suppose we have N multiplets (thus, $N = N_d$) and let m_j, ξ_j be mass and non-minimal coupling of the jth multiplet. Constraints (2.3) in such a theory are satisfied and the induced cosmological constant vanishes. Relations (2.4), (2.6) can be written in the form

$$\sum_{j=1}^{N} x_j = 0\,, \qquad \sum_{j=1}^{N} x_j m_j^2 = 0, \qquad \sum_{j=1}^{N} x_j m_j^2 \ln m_j^2 = \frac{\pi}{G}\,, \tag{A.1}$$

where $x_j = 1/2 - 2\xi_j$. One can consider (A.1) as a system of equations which defines parameters x_j at given masses m_j and a given value G. The system is linear and allows solutions for $N \geq 3$.

The solution is easy to find for $N = 3$ and to get explicit expressions for 2D gravitational coupling $G_2(p)$ and the running 4D coupling $G(p)$, see (2.21), (2.24). After some algebra the results can be represented in a simple form

$$\frac{G}{G(p)} = \frac{D(p)}{D(0)}\,, \qquad \frac{G_2(0)}{G_2(p)} = \frac{D'(p)}{D'(0)}\,, \tag{A.2}$$

$$D(p) = f(\mu_1(p), \mu_2(p), \mu_3(p))\,, \tag{A.3}$$

$$f(a,b,c) = (c-b)a\ln a + (a-c)b\ln b + (b-a)c\ln c\,, \tag{A.4}$$

$$D'(p) = \frac{\partial}{\partial p^2} D(p) = g(\mu_1(p), \mu_2(p), \mu_3(p))\,, \tag{A.5}$$

$$g(a,b,c) = (c-b)\ln a + (a-c)\ln b + (b-a)\ln c\,, \tag{A.6}$$

where $\mu_k(p) = m_k^2 + p^2$ and m_1, m_2, m_3 are the masses of the three multiplets. The 2D coupling is defined as

$$\frac{1}{G_2(0)} = -\frac{\partial}{\partial p^2}\frac{1}{G(p)}\bigg|_{p=0}\,. \tag{A.7}$$

It can be shown that if $G > 0$ than both functions $G(p)$, $G_2(p)$ are positive and monotonically increasing as parameter p increases.

B. Black hole entropy in 2+1 dimensions

In this appendix we demonstrate that our derivation of the black hole entropy works for space-times where the number of dimensions is different from four. As an example, we consider a theory in a three-dimensional space-time. This case is instructive because quantum theories in odd and even dimensions have different properties. It is also interesting because the entropy of some three-dimensional black holes, such as the BTZ (Banados-Teitelboim-Zanelli) black hole [8], allows statistical-mechanical representation by methods of conformal theory [7].

Induced gravity models in three dimensions were discussed in [25]. For the model consisting of spinor and non-minimally coupled scalar constituents the constraints which ensure finiteness of the induced Newton constant can be written in the form (4.2). Central charges of scalar fields are $c_s = 1 - 6\xi_s$, while charges of spinor fields are $c_d = 1$ in accord with the fact that spinors have 2 components in $D = 3$. Given these constraints the induced Newton constant is defined by the relation [25]

$$\frac{1}{G} = -\frac{1}{3}\sum_k c_k m_k\,, \tag{B.1}$$

where the parameters ξ_s are chosen so that $G > 0$. The horizon of a black hole in three dimensions is a circle of some length a [8] and the corresponding Bekenstein-Hawking entropy is

$$S^{BH} = \frac{a}{4G} \, . \tag{B.2}$$

We now use the same line of arguments as in four dimensions and consider entropy of constituents in the near-horizon limit. In the limit when a is large the total entropy of 2D constituents at the momentum p coincides with (4.4)

$$s(p) = \sum_k s\left(c_k, \frac{R_k(p)}{\epsilon}\right) = \frac{1}{6} \sum_k c_k \ln R_k(p) \, . \tag{B.3}$$

The entropy of the 3D theory is

$$S = \frac{a}{\pi} \int_0^\infty s(p) dp \ . \tag{B.4}$$

By performing the integration in (B.4) one finds the expression

$$S = -\frac{a}{12} \sum_k c_k m_k \ , \tag{B.5}$$

which is finite because of the induced gravity constraints and coincides with the Bekenstein-Hawking entropy (B.2).

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